EMJ MS

EUROPEAN MULTIDISCIPLINARY JOURNAL OF MODERN SCIENCE

https://emjms.academicjournal.io/index.php/ Volume: 4

On a Qualitative Picture of a Homogeneous Two-Dimensional System's Trajectory

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Abstract: The article considers the global picture of the trajectories of a two-dimensional system

$$\frac{dx}{dt} = X_i^m(x), \; x = (x_1, x_2),$$

Here - is homogeneous continuous functions of degree of homogeneity having an isolated O at the beginning.

Keywords: global picture trajectory, isolated zero, classifications of integral manifolds, and - limit sets of trajectories, singular points, characteristic number, integral sector, parabolic sector, hyperbolic sector, elliptic sector, periodic trajectories, stable, unstable, generalized knot, generalized node.

The paper considers the global (as a whole) picture of the trajectories of a two-dimensional system

$$\frac{dx}{dt} = X_i^m(x), \; x = (x_1, x_2), \quad (1)$$

Where $X_i^m(x), (i = 1, 2)$ - homogeneous continuous functions of degree of homogeneity $m \geq 1,$ having an isolated *O* at the beginning

System (1) was considered by H. Forster, G.E. Shilov, N.P. Erugin and others. The disadvantages of these methods is that they are not applicable to *n*-dimensional homogeneous system $n \geq 3.$ We base the classification of integral manifolds of system (1) on the structure ω and α - are limit sets of trajectories of a circle centered at the origin, radius unity. This method can be extended to -dimensional homogeneous system.

Let $x_1 = u_i x_2$ then system (1) has the form

$$\left.\begin{aligned} &a)\ \frac{du}{d\tau}=U(u) \\ &б)\ \frac{dr}{d\tau}=rR(u) \end{aligned}\right\}, \ (2)$$

Where $U(u)=X^m(u)-uR(u),\ R(u)=u_1X_1^m(u)+u_2X_2^m(u)$

On the sphere S^1 ω -limit set Ω_γ. Trajectory γ:

1) Consists of one single point g, which is special, i.e.. $\Omega_\gamma=g$ or

2) Consists of a single periodic trajectory S^1, т.е. $\Omega_\gamma=S^1$.

In this case, the set Ω_γ will be done α - limit set has the same structure as Ω_γ.

On a circle S^1 with isolated singular points, only three types of trajectories are possible:

a) Single point g^i, b) trajectory γ, approaching to singular point g, c) closed trajectories θ coinciding with S^1.

Consider the sector $C(\gamma)$, образованный rays passing through point *O* and trajectory points γ of a system (2) a). This sector is an integral manifold of system (1) and, consequently, of systems (1). Plane $\mathbb{R}^2$ is divided into sets of mutually non-intersecting integral sectors, on each of which one can study the picture of the trajectories of system (1).

Everywhere we denote by γ and ω trajectories of systems (2.1.2a) and (2.1.1), respectively. Obviously, $\omega=r(\gamma)\gamma,$ (3)

Where $r(\gamma)$ - solution of equation (1.7.2b) defined by the formula

$$r(\gamma)=r_0\exp\int_{\tau_0}^{\tau}R(\gamma)d\tau, \ (4)$$

Trajectories of system (2.1.1) have the following properties:

1) If the beam $C(g)\in C(\gamma)$ are integral, then by virtue of the uniqueness of the solutions, but one of the trajectories $\omega\in C(\gamma)$ does not intersect this ray.

2) If the trajectory $\omega\in C(\gamma)$ intersect each non-integral ray in the same direction and form the same non-zero angle with it.

Let's turn to the consideration of the global picture of the trajectories of system (1) on the integral sectors corresponding to three types of trajectories.

a) Isolated points g^i. Let g^i - isolated circle points S^1.

Determination. Number $R(g^i)$ is called the characteristic number of isolated points g^i.

If the trajectory γ is an isolated point of the system (2a), then the integral sector $C(\gamma)$ degenerates into an integral ray $\lambda g,\ \lambda > 0.$ These integral rays (called exceptional directions or characteristic directions or invariant rays) determine by means of real solutions of the algebraic equation.

$$\frac{x_1}{X_1^m(x)} = \frac{x_2}{X_2^m(x)}, \quad (5)$$

Note that the two-dimensional system (1) does not have an integral ray, and then the singular point will be a singular point of the focus or center type, which will be discussed below.

b) Trajectories adjoining singular points. On the sphere S^1 trajectory γ adjoins to only two distinct singular points.

If γ will adjoins to two singular points g^1 и g^2 and that $\Omega_\gamma = g^1,\ A_\gamma = g^2,\ (g^1 \neq g^2).$ the sector boundary is a point $C(\gamma)$ O and integral rays $C(g^1)$ and which, being one-dimensional integral manifolds, have a consistent direction along $\tau.$

Ray points $C(g^i)$ $(i = 1,2)$ cannot belong to limit sets Ω_γ и $A_\omega,$ of any trajectories $\omega.$ Since at $\gamma \to g^1$ and $\tau \to +\infty$ and $\gamma \to g^2$ при $\tau \to -\infty,$ then each of the limit sets consists of a point O or *B-U* (infinitely distant) points in the direction g^1 at $\tau \to +\infty,$ and in the direction g^2 at $\tau \to -\infty.$ Depending on the signs $R(g^1)$ and $R(g^2)$ on the sector $C(\gamma).$ System trajectories can be arranged in the following ways (2):

1) Parabolic sector at $R(g^1)R(g^2) > 0;$

2) Hyperbolic sector at $R(g^1) > 0,\ R(g^2) < 0;$

3) Elliptical sector at $R(g^1) < 0,\ R(g^2) < 0;$

в) Periodic trajectory $\theta = S^1.$

For a two-dimensional homogeneous system. The periodic solution of system (2a) is unique $\theta = S^1$ and if $\nabla(\theta) = \int_0^{2\pi} R(\theta(\tau))d\tau = 0,$ then on the basis of Lemma 3 of Section 1.4, the origin of the system (1) will be a singular point of type center. If $\nabla(\theta) = \int_0^{2\pi} R(\theta(\tau))d\tau \neq 0,$ then the origin will be a singular point of the focus type. Focus will be stable at $\nabla(\theta) < 0$ and unstable at $\nabla(\theta) > 0.$

If points a), b) prevent system (2a) from having a periodic solution, then the following theorem holds:

Theorem. It is essential and sufficient for all numbers to have negative (positive) signs in order for the origin of system (1) to be a stable (unstable) generalized node. $R\left(g^{i}\right)$

It is required and sufficient for the origin of system (1) to be a generalized saddle if at least one pair of numbers $R\left(g^{i}\right)$ has different signs.

2) On one-of-a-kind trajectories.

Integral rays $C\left(g^{i}\right)$ have the potential to be unique. Formally, these rays can be derived from (2.1.4) at $r_0 = 0,$ it $\int_{\tau_0}^{+\infty} R(u)d\tau = +\infty$ and at $r_0 = \infty,$ if $\int_{\tau_0}^{\tau} R(u)d\tau = -\infty$ and in these cases we will not call them singular trajectories.

We write in the form to detect peculiar trajectories of system (1) of one of the coordinates of the point $g(u_1, u_2) \in S^1$, as well as the parameter and solution of equation (2a).

$$r(\gamma) = r_0 exr\left[\int_{u_1^0}^{u_1} \frac{R(u)du_1}{X_1^m(u) - u_1 R(u)}\right]. \quad (4)$$

Since $\lim\limits_{u_1 \to u_1^1} J = \lim\limits_{u_1 \to u_1^1} \int_{u_1^0}^{u_1} \frac{R(u)du_1}{X_1^m(u) - u_1 R(u)} = b$ finite number, then the trajectory is $C\left(g^1\right),$ special, since it cannot be obtained from (4) for a particular value of the constant $r_0,$ because in here $r_0 = re^{-b}$ function r of at the points $C\left(g^1\right),$ of the beam $g\left(u_1^1, u_2^1\right)$ where is an isolated point $S^1.$

Therefore, some of the trajectories of the half-lines $C\left(g^{i}\right)$ may be special. If equation (5) has no real solutions, then system (1) $C\left(g^{i}\right)$ has no special trajectories.

It can be shown, as in Section 1.6, that in a sector with boundaries at a point and integral rays $C\left(g^{k}\right)$ and $C\left(g^{k-1}\right),$ if $J \to b$ is $u_1 \to u_1^1,$ then the trajectory passing through an arbitrary point $\left(x_1^0, x_2^0\right)$ which is not lying on the integral rays $C\left(g^{k}\right),$ $C\left(g^{k-1}\right)$ intersected by integral beam $C\left(g^{k}\right)$ at the point $\left(x_1^1, x_2^1\right).$

Here we see that if $u_1 \to u_1^1,$ J has a finite limit, then the half-line $C\left(g^{k}\right)$ will be a special trajectory and will be passed through the point $\left(x_1^0, x_2^0\right) \in G(\gamma),$ limited by integral beams $C\left(g^{k}\right)$ и $C\left(g^{k-1}\right),$ enters a certain point $\left(x_1^1, x_2^1\right) \subset C\left(g^{k}\right).$ Thus, several trajectories of system (1) pass through this point. Note that this trajectory cannot return to the sector $C(\gamma).$

Indeed, from $x_i = x_i^1 exr\left[\int_{u_1^0}^{u_1} \frac{R(u)du_1}{X_1^m(u) - u_1 R(u)}\right]$ single-valued functions of the parameter

u_1, therefore, due to property 2, this trajectory intersects with each ray *l* at one point.

Remark 1. If *O* the non-isolated singular point of the system (1), then the degree of single-phase decreases. $R(u)=0,$ then $x^m(u)=0.$

Let $\frac{x_1}{x_2}=\tau=\tau_0$ is an k - multiple solution of equation (4), then $x_i^m(x)$ can be written in the form $x_i^m=\left(x_1-\tau^0 x_2\right)^k X_i^{m-k}(x)$ and system (1) can be given the form

$$\frac{dx_i}{dt}=\left(x_1-\tau^0 x_2\right)^k X_i^{m-k}(x).$$

Here, introducing a new time, setting $\left(x_1-\tau^0 x_2\right)^k dt=dt_1,$ we get $\frac{dx_1}{dt_1}=X_i^{m-k}(x),$ where $k\geq 1.$

Remark 2. Since the sphere S^1 is is filled with singular points, then system (1) reduces to the form $\frac{dx}{dt}=Ex,$ where $E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$ for which the singular point *O* is a node.

Example 1: Consider the system

$$\left.\begin{aligned}\frac{dx_1}{dt}&=a_{30}x_1^3+a_{21}x_1^2x_2+a_{12}x_1x_2^2+a_{03}x_2^3\\ \frac{dx_2}{dt}&=a_{21}x_1^3+a_{12}x_1^2x_2+a_{03}x_1x_2^2+b_{03}x_2^3\end{aligned}\right\}\quad(6)$$

System (2) takes the form

$$\left.\begin{aligned}\frac{du_1}{d\tau}&=a_{30}u_1^3+a_{21}u_1^2u_2+a_{12}u_1u_2^2+a_{03}u_2^2-u_1R(u)\\ \frac{du_2}{d\tau}&=a_{12}u_1^3+a_{21}u_1^2u_2+a_{03}u_1u_2^2+b_{03}u_2^2-u_2R(u)\\ \frac{dr}{d\tau}&=rR(u)\end{aligned}\right\}$$

Where $r^2dt=d\tau,\ R(u)=a_{31}u_1^4+2a_{12}u_1^3u_2+2a_{21}u_1^2u_2^2+2a_{03}u_1u_2^2+b_{03}u_2^4.$

There are four singular points S^1 on a sphere (circle).

$$g^{1-4}\left(\pm\frac{k_i}{\sqrt{1+k_i^2}},\pm\frac{1}{\sqrt{1+k_i^2}}\right),\text{где } k=\pm\sqrt[4]{\frac{a_{30}}{b_{03}}}\ \text{при}\ a_{30}b_{03}>0,\ i=1,2.$$

$$R\left(g^{1-2}\right)=\frac{a_{30}^2}{b_{03}}+2a_{12}\sqrt[4]{\left(\frac{a_{30}}{b_{03}}\right)^3}+2a_{12}\sqrt{\frac{a_{30}}{b_{03}}}+2a_{03}\sqrt[4]{\frac{a_{30}}{b_{03}}}+b_{03},$$

$$R\left(g^{3-4}\right)=\frac{a_{30}^2}{b_{03}}-2a_{12}\sqrt[4]{\left(\frac{a_{30}}{b_{03}}\right)^3}+2a_{12}\sqrt{\frac{a_{30}}{b_{03}}}-2a_{03}\sqrt[4]{\frac{a_{30}}{b_{03}}}+b_{03}.$$

By virtue of the previous theorems, if $R\left(g^{1-2}\right)R\left(g^{3-4}\right)>0,$ then the origin of system (1) will be a generalized node, and if $R\left(g^{1-2}\right)R\left(g^{3-4}\right)>0$ - it will be a generalized saddle.

For $a_{30}b_{03}<0$ there are no singular points on the circle S^1 and we set $a_{30}=-b_{03},$ $a_{12}=a_{03}, a_{12}=0,$ then system (1.7.6) takes this form

$$\left.\begin{aligned}\frac{dx_1}{dt}&=a_{30}x_1^3+a_{21}x_1^2x_2+a_{12}x_2^3\\ \frac{dx_2}{dt}&=a_{21}x_1^3+a_{12}x_1x_2^2-a_{03}x_2^3\end{aligned}\right\}$$

Calculations show that in order for the origin of coordinates of this system to be the center, it is necessary and sufficient $a_{12}=a_{30}$.Since $a_{12}\neq a_{30},$ then the origin of coordinates *O* will be a focus, and a stable focus at and unstable focus at $a_{12}<a_{30}.$

Let's look at a few examples.

Example 2

$$\left.\begin{aligned}\frac{dx_1}{dt}&=ax_1+bx_2\\ \frac{dx_2}{dt}&=-bx_1+ax_2\\ \frac{dx_3}{dt}&=cx_3\end{aligned}\right\},\ a\neq c.\ (7)$$

S^1 has two singular points on a sphere $g^{1,2}\left(0,0,\pm1\right)$ and one periodic solution $u_3=0.$ For these special points $R\left(g\right)$ and $\nabla\left(\theta\right)$ will be equal:

$$R\left(g^{1,2}\right)=c,\ \ \nabla\left(\theta\right)=\frac{2\pi a}{|b|}.$$

Therefore, if $ac>0$ the singular point of system (7) is a focus (generalized node), if $ac<0$ saddle focus (generalized saddle).

Example 3. $\frac{dx}{dt}=\lambda_i X_i^{2m+1};\ \ i=\overline{1,3},\ \lambda_i\neq 0.$

If all λ_i are positive (negative), then we will have 26 singular points on the sphere S^2, the coordinates of which are:

$$g^{1,2}(0,0,\pm1),\ g^{3,4}(0,\pm1,0),\ g^{5,6}(\pm1,0,0),\ g^{7,8}\left(0,B,\pm\sqrt[2m]{\frac{\lambda_2}{\lambda_3}}B\right),$$

$$g^{9,10}\left(0,-B,\pm\sqrt[2m]{\frac{\lambda_2}{\lambda_3}}B\right),\ g^{11,12}\left(\pm\sqrt{\frac{\lambda_3}{\lambda_1}}C,0,C\right),\ g^{13,14}\left(\pm\sqrt{\frac{\lambda_3}{\lambda_2}}C,0,-C\right),$$

$$g^{15,16}\left(\pm\sqrt{\frac{\lambda_2}{\lambda_2}},D,0\right),-g^{17,18}\left(\pm\sqrt{\frac{\lambda_2}{\lambda_2}}D,D,0\right),-g^{19,20}\left(\pm\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},\pm\frac{1}{\sqrt{A}}\right),$$

$$-g^{21,22}\left(-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},\pm\frac{1}{\sqrt{A}}\right),-g^{23,24}\left(-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_2}}}{\sqrt{A}},\pm\frac{1}{\sqrt{A}}\right),$$

$$g^{25,26}\left(-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},-\frac{\sqrt[2m]{\frac{\lambda_3}{\lambda_1}}}{\sqrt{A}},\pm\frac{1}{\sqrt{A}}\right),$$

Here $A=1\pm\sqrt[m]{\frac{\lambda_3}{\lambda_1}}+\sqrt[m]{\frac{\lambda_3}{\lambda_2}},\ B=\sqrt{\frac{1}{1+\sqrt[m]{\frac{\lambda_2}{\lambda_3}}}},\ C=\sqrt{\frac{1}{1+\sqrt[m]{\frac{\lambda_3}{\lambda_2}}}},\ D=\sqrt{\frac{1}{1+\sqrt[m]{\frac{\lambda_2}{\lambda_1}}}}.$

If λ_i have different signs, then g^{7-26} only four remain. The sphere has ten singular points. Obviously, there are no periodic solutions. $R(g^i)$ will be equal:

$$R(g^{1,2})=\lambda_3,\ R(g^{3,4})=\lambda_2,\ R(g^{5,6})=\lambda_1,\ R(g^{7,8,9,10})=\frac{\lambda_2}{\left(1+\sqrt[2m]{\left(\frac{\lambda_2}{\lambda_3}\right)^2}\right)^m},$$

$$R(g^{11,12,13,14})=\frac{\lambda_3}{\left(1+\sqrt[2m]{\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right)^m},R(g^{15,16,17,18})=\frac{\lambda_2}{\left(1+\sqrt[2m]{\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right)^m},\ R(g^{19-26})=\frac{\lambda_3}{A^m}.$$

It is easy to see that if all $\lambda_i > 0(\lambda_i < 0)$, then the origin of system (8) will be a node, if λ_i have different signs, then the beginning will be a saddle.

Example 4. Consider a triangular system (A.A. Shestakov)

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_1 x_1^{2m+1} \\ \frac{dx_2}{dt} &= a_2 x_1^{2m+1} + b_2 x_2^{2m+1} \\ \frac{dx_3}{dt} &= a_3 x_1^{2m+1} + b_3 x_2^{2m+1} + c_3 x_3^{2m+1} \end{aligned}\right\}, \quad (9)$$

Auxiliary system (2a) takes the form

$$\left.\begin{aligned} \frac{du_1}{dt} &= u_1\left[a_1 u_1^{2m} - R(u)\right] \\ \frac{dx_2}{dt} &= a_2 u_1^{2m+1} + b_2 u_2^{2m+1} - u_2 R(u) \\ \frac{dx_3}{dt} &= a_3 u_1^{2m+1} + b_3 u_2^{2m+1} + c_3 u_3^{2m+1} - u_3 R(u) \end{aligned}\right\}, \quad (10)$$

Where $R(u) = a_1 u_1^{2m+2} + c u_3^{2m+2} + (a_2 u_2 + a_3 u_3) u_1^{2m+1} + (b_2 u_2 + b_3 u_3) u_2^{2m+1}$.

This system has the following singular points:

$$(0,0,\pm 1),(0,\pm kA, A)(0,-kA,-A),(vz\alpha, v\alpha, z\alpha),\ (-vz\alpha,-v\alpha,-z\alpha),$$

Where $A = \frac{1}{\sqrt{1+k^2}},\ \alpha = \frac{1}{\sqrt{v^2 z^2 + v^2 + z^2}},\ a, k, v, z$ - respectively, the roots of the equation

$$b_3 k^{2m+1} - b_2 k^{2m} + c_3 = 0,\ a_2 z^{2m+1} - a_1 z^{2m} + b_2 = 0,$$

$$\left(a_2 b_3 - a_3 b_2 - \frac{a_1}{z}\right) v^{2m-1} + a_1 v^{2m} - c_3 = 0.$$

System (2.1.10) has no periodic solutions, but $R(g^i)$ will be equal to:

$$R(0,0,\pm 1) = c_3,\ R(0,kA,A) = R(0,-kA,-A) = b_2 A^{2m+2} k^{2m}(k^2+1),$$

$$R(vz\alpha, v\alpha, z\alpha) = R(-vz\alpha,-v\alpha,-z\alpha) = a_1 z^{2m+2} v^{2m}(2v^2+1).$$

If here $a_1 < 0,\ b_2 < 0,\ c_3 < 0,$ then all trajectories of system (1.7.4) are O^+-curves (positive node) and if $a_1 > 0,\ b_2 > 0,\ c_3 > 0,$ then all trajectories of system O^-- curves (negative node).

Notes

1. The singular point of the *o* system (1) is unstable, if $X^m(x)$ has the property. $X^m(-x)=X^m(x)$. (9). Under condition (10), the functions $U(u)$ and $R(u)$ has the property. $U(-u)=U(u),\ R(-u)=-R(u)$ (10)

Let the points $g(\pm u^0)\in S^2$ be special. Due to property (10), the trajectories ω on $C(g(\pm u^0))$ are defined by the formula

$\omega = g(\pm u^0)r_0\exp\left[\pm R(u^0)(\tau-\tau_0)\right]$ (12)

a) Since $R(u^0)>0$ and $\tau\to+\infty$ (or $R(u^0)<0$ and $\tau\to-\infty$), then $\omega\to+\infty$ however $C(g(u^0))$, a $\omega\to 0$ at the $C(g(-u^0))$;

b) Since $R(u^0)<0$ and $\tau\to\infty$(or $R(u^0)>0$ and $\tau\to-\infty$), then $\omega\to 0$ at $C(g(u^0))$, but $\omega\to+\infty$ at $C(g(-u^0))$.

On the integrated beam $G(g):\{C(g(u^0))\cup C(g(-u^0))\}$ trivial solution of O-unstable.

Since system (1) has (at least one) integral ray, hence the singular point O - unstable.

2. Under condition (9), O-we will call it a generalized knot if all trajectories are O- curves. Under condition (9), the generalized node cannot be positive (negative).

3. Note 1 holds for a homogeneous system of arbitrary dimension with property (9).

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