EMJ MS

EUROPEAN MULTIDISCIPLINARY JOURNAL OF MODERN SCIENCE

https://emjms.academicjournal.io/index.php/ Volume:4

On the Second-Degree Homogeneous Differential System's Exclusive Directions

Khusanov Bozorboy

Candidate of Physical and Mathematical Sciences, Samarkand State Institute of Architecture and Civil Engineering, Uzbekistan

Kulmirzaeva Gulrabo Abduganiyevna

Senior teacher of Samarkand State Institute of Architecture and Civil Engineering, Uzbekistan

Abstract: In the article we study the case of coexistence of exceptional directions of the system.
dx/dt=X^2 (x), x = (x1 , x2 , x3),
Here X2(x) – is homogeneous polynomial vector is a function of degree 2, and X_3^2(x) has a factor x3 .

Keywords: polynomial vector-function, vector norm, equator, manifold, isolated, saddle, knot, diametrically.

In this paper, we consider the behavior of the trajectory of a homogeneous polynomial differential system of the second degree

$$\frac{dx}{dt} = X^2(x),\ x = (x_1, x_2, x_3)\ (1)$$

Here X^2 (x) – homogeneous polynomial vector - function of degree 2. Let us study the cases of coexistence of exceptional directions of the system (1), where $X_3^2(x)$ has a multiplier x_3.

The study will be carried out by the method of work [1].

Introducing into system (1), we introduce the substitution

$x = ru_i\ (i = \overline{1,3}\ )$ (2)

where u = (u_1, u_2 , u_3) – unit vector, *r* – the norm of the vector *x*, we get

$$\left.\begin{aligned} a)\ \frac{du}{dt_1} &= X^2(u) - uR(u) \\ б)\ \frac{dr}{dt_1} &= rR(u) \end{aligned}\right\}\ (3)$$

Where $R(u) = X^2(u)\,u,\ dt_1 = rdt.$ (3a)

On the basis of [1], singular points of the sphere S2 or system (3a) correspond to the exceptional directions of system (1), and singular points lying on the equator $u_1^2 + u_2^2 = 1$ system (3a) , correspond to the exceptional directions of the integral manifold $x_3 = 0$.

Substituting $u_3 = 0$ in the system (3a) and then , entering $u_1 = \mu u_2$ (or $u_2 = \mu u_1$), we get :

$$\frac{d\mu}{dt_1} = \frac{X_1^2(\mu,1) - \mu X_2^2(\mu,1)}{u_2[X_2^2(\mu,1) - u_2^2 R(\mu,1)]},\ (4)$$

Coordinates of special points of the equator $u_1^2 + u_2^2 = 1$ spheres S^2

will look like.

$$\left(\frac{\mu_0}{\sqrt{1+\mu_0^2}}, \frac{1}{\sqrt{1+\mu_0^2}}, 0\right), \left(-\frac{\mu_0}{\sqrt{1+\mu_0^2}}, -\frac{1}{\sqrt{1+\mu_0^2}}, 0\right)$$

where μ_0 – real root of the equation.

$$X_1^2(\mu,1) - \mu X_2^2(\mu,1) = 0, X_2^2(\mu_0,1) \neq 0 \quad (5)$$

The nature of these singular points determines the nature of the exceptional directions of the manifold $x_3 = 0$.

To study the exceptional directions of system (1) not lying on the manifold

$x_3 = 0$ (for which $u_3 \neq 0$), in a system (3a) we conduct substitutions

$y = \frac{u_1}{u_3}, z = \frac{u_2}{u_3}$ then she will look like:

$$\left.\begin{array}{l} a) \begin{cases} \frac{dy}{d\tau} = X_1^2(y,z,1) - yX_3^1(y,z,1) = \Phi(y,z) \\ \frac{dz}{d\tau} = X_2^2(y,z,1) - zX_3^1(y,z,1) = \psi(y,z) \end{cases} \\ б) \ \frac{du_3}{d\tau} = u_3[X_3^1(y,z,1) - u_3^2 R(y,z,1)] \end{array}\right\} (6)$$

Singular points of system (6) determined from the solutions of the system.

$$\left.\begin{array}{r} u_3 = 0 \\ \Phi(y,z) = 0 \\ \psi(y,z) = 0 \end{array}\right\} (7)$$

Correspond to the exceptional directions of system (1) that do not lie on the manifold $x_3 = 0$. The singular points determined from the solution of the system

$$\left.\begin{array}{r} X_3^1(y,z,1) = 0 \\ X_2^2(y,z,1) = 0 \\ X_1^2(y,z,1) = 0 \end{array}\right\} (8)$$

Correspond to the singular lines of system (1) (in this case, the singular point 0 is not isolated and we will not consider it).

System (7) may have four , three, two, one or no solution, therefore, system (1) may have four, three, two, one or may not have exceptional directed , not lying on the manifold $x_3 = 0$.

1°. Exceptional directions of system (1) corresponding to singular points determined from system (7) or from equation (5), or corresponding to singular points of the first group of the sphere $S^{2,}$ we will call exceptional directions of type I or type II, or the first group, respectively.

Lemma 1. Exceptional directions of type II can only be of the first group.

Proof. All points of the equator $u_1^2 + u_2^2 = 1$ of sphere S^2 are singular points of the first group, since the equator $u_3 = 0$ is the solution.

Lemma 2. Singular points of the equator of the sphere S^2, $u_1^2 + u_2^2 = 1$ can only be nodes, saddles, or open saddle nodes.

Proof. . Let $\mu = \mu_0$ be k - multiple real root of equation (5) then, introducing the substitution $\mu - \mu 0 = \bar{\mu}$ into differential equation (4) , we will have the Brno - Bouquet equation [2]

$$\frac{d\bar{\mu}}{du_2} = \frac{[X_1^2(\mu,1) - \mu X_2^2(\mu,1)]_{\mu=\mu_0}^{(k)} \mu^{-k} + \cdots}{u_2[X_2^2(\mu_0,1) + \cdots]}$$

where k = $\overline{1,3}$, $X_2^2(\mu_0, 1) \neq 0$. For it, for even k, the origin of coordinates $u_3 = 0, \mu = 0$ there will be an open saddle - a node if k is odd, and a node (saddle) when

$$[X_1^2(\mu, 1) - \mu X_2^2(\mu, 1)]^{(k)}_{\mu=\mu_0} X_2^2(\mu_0, 1) > 0\ (< 0)$$

Therefore, the singular points of the equator $u_1^2 + u_2^2 = 1$ of sphere S^2 there will also be only nodes, saddles or open saddles - nodes.

Lemma 3. System (1) may not have exceptional directed to the manifold x3 = 0 if the identity holds.

Proof. Indeed, when identity (9) is satisfied, splitting (5) is fulfilled identically and on the equator

$u_1^2 + u_2^2 = 1$ Spheres S^2 there are no singular points, therefore, there are no exceptional directions to the manifold x3 of system (1).

Lemma 4. System (3a) at the equator $u_1^2 + u_2^2 = 1$ of sphere S^2 can have six, four, two singular points (two diametrically opposite), if D < 0, D = 0, D > 0 respectively, here

$$D = [108\left(\frac{b_{110} - a_{200}}{b_{200}}\right)^2 - \frac{2}{3}\left(\frac{b_{110} - a_{200}}{b_{200}}\right)\left(\frac{b_{020} - a_{110}}{b_{200}}\right) - 2\frac{a_{020}}{b_{200}}]^2 + [-\left(\frac{b_{110} - a_{200}}{b_{200}}\right)^2 + + \frac{3(b_{020} - a_{110})}{b_{200}}]^3$$

where a_{ijk}, b_{ijk} - coefficients of homogeneous polynomials $X_1^2(x), X_2^2(x)$ accordingly.

Proof. It follows from the fact that equation (5) is presented in the form

$$b_{200}\mu^3 + (b_{110} - a_{200})\mu^2 + (b_{200} - a_{110})\mu - a_{020} = 0$$

Last at D < 0 have 3 at D = 0 two (with one double), and at D > 0 one real solution: therefore, system (3a) in the case D < 0 have 6, when D = 0 four and in case D > 0 two (diametral opposite) singular points.

Lemma 5. If system (3a) has six (two diametrically opposite) singular points on the equator of the sphere S^2, then all six cannot be of the saddle type.

Proof. Similar to the proof of Lemma 2 in [3].

Lemma 6. If (9) identity holds, then system (1) cannot have four exceptional directions of type I .

Proof. When identity (9) is fulfilled, system (7) can be written as**:**

$$\left.\begin{aligned} \Phi(y,z) &= a_0 + a_1 y + a_2 z + y f_1(y,z) \\ \psi(y,z) &= b_0 + b_1 y + b_2 z + z f_2(y,z) \end{aligned}\right\}$$

As we know [4], such a system cannot have four solutions; therefore, system (1) when identity (9) is satisfied does not have four exceptional directions of type I.

2°. Let system (1) have four exceptional directions of type I , then it can be reduced to the form with the help of an unexpressed affine transformation.

$$\left.\begin{aligned} \frac{dx_1}{dt} &= (1 + a_1)X_1^2 + \left(\frac{1-\alpha}{\beta} + \frac{1-\beta}{\alpha}c + b_1\right)x_1x_2 + c_1X_2^2 + (c_1 - 1)x_1x_3 - cx_2x_3 \\ \frac{dx_2}{dt} &= k[X_1^2 - x_1x_3 + (a - b_1)X_1^2 - (a - c_1)x_1x_3 + \left(\frac{1-\alpha}{\beta} + \frac{1-\beta}{\alpha}a + a_1\right)x_1x_2] \\ \frac{dx_3}{dt} &= a_1x_1x_3 + b_1x_2x_3 + c_1X_3^2 \end{aligned}\right\}$$

System (6a) then takes the form of an equation.

$$\frac{dz}{dy} = k\frac{y(y-1) + az(z-1) + \left(\frac{1-\alpha}{\beta} + \frac{1-\beta}{\alpha}a\right)yz}{y(y-1) + cz(z-1) + \left(\frac{1-\alpha}{\beta} + \frac{1-\beta}{\alpha}c\right)yz}$$

Each singular point of which corresponds to the upper parts of the exceptional directions of type I of system (1). The hemisphere S^2 has the following singular points

A(0, 0, 1), B(0, 1, $\frac{1}{\sqrt{2}}$), C(1, 0, $\frac{1}{\sqrt{2}}$), E(α, β, $\frac{1}{\sqrt{\alpha^2+\beta^2}}$), $F_i(\frac{\mu_i}{\sqrt{\alpha^2+\beta^2}}, \frac{1}{\sqrt{\alpha^2+\beta^2}}, 0)$

where $i = \overline{1,3}$.

Lemma 7

a) If D < 0, 1 - α – β > 0, α > 0, β > 0 (or1 – α – β < 0,

$\alpha \cdot \beta < 0$), then on the hemisphere S^2 two of the singular points A, B, C, E will be anti saddles , the other two are saddles , and two of the singular points F_i will be nodes ;

b) If D > 0, 1 – $\alpha - \beta > 0, \alpha > 0, \beta < 0$ (or 1- $\alpha - \beta < 0 , \alpha \cdot \beta < 0$) , then on the hemisphere S2 two of the singular points A, B, C, E will be antisaddles , and of the singular points of F_i two merge, and the double singular point

$F_1 = F_2$ will be open, F_3 – node;

c) If D > 0, 1- α – β > 0, α > 0, β > 0 (or 1 – α –β < 0, $\alpha \cdot \beta < 0$), then two of the singular points F_i disappear, one will be a node.

Proof. On the S^2 the sum of the singular point indices is equal to 2, and on the hemisphere it is equal to 1 [5]. Considering also that if 1 – $\alpha - \beta > 0, \alpha > 0, \beta > 0$ or 1- $\alpha - \beta < 0 , \alpha \cdot \beta < 0$, then of the four singular points A, B, C, E will be anti saddles, and the other two saddles, with D < 0 we have a case

a) at D = 0 happening

b) at D > 0 case ,

c) distribution of singular points of the semi-equator of the sphere S^2.

Lemma 8. a) If 1 – $\alpha - \beta < 0, \alpha > 0, \beta > 0, k(a-c) < 0$ or

1 – $\alpha - \beta > 0, \alpha < 0, \beta < 0 , k(a-c) > 0$ or 1 – $\alpha - \beta > 0$,

$\alpha \cdot \beta < 0$), k(a – c) < 0 , D > 0.

b) Under condition a) on the hemisphere S^2 of the four singular points A, B, C, E, three saddles and an anti saddle, on the semi-equator $u_1^2 + u_2^2 = 1$ – three knots.

The number of exceptional directions of the system (1) corresponding to singular points such as anti-saddle , saddle, opening saddle - node of the sphere S^2 will be denoted by N(a) , N(c) and N(cy) .

The lemmas that have been proved imply:

Theorem 1. a)If 1 - α – β > 0, α > 0, β > 0 (or 1 – α – β < 0,

$\alpha \cdot \beta < 0$) D < 0, then N(a) = 4, N(c) = 3.

b) D = 0, то N(a) = 3, N(c) = 2, N(cy) = 1

c) D > 0 , то N(a) = 3, N(c) = 2

Theorem 2. a) If $1 - \alpha - \beta > 0$, $\alpha \cdot \beta < 0$ ($1 - \alpha - \beta > 0$, $\alpha < 0$, $\beta < 0$)

$D = 0$, we have $N(a) = 3$, $N(c) = 2$, $N(cy) = 1$

б) $D > 0$, in this situation will be $N(a) = 3$, $N(c) = 2$.

Theorem 3. If $1 - \alpha - \beta < 0$, $\alpha > 0, \beta > 0$ *or* $1 - \alpha - \beta < 0$, $\alpha < 0$, $\beta < 0$ *or* $1 - \alpha - \beta > 0$, $\alpha \cdot \beta < 0$, we will have $N(a) = 4$, $N(c) = 3$.

References

1. Sharipov Sh.R. "Classification of integral manifolds of a homogeneous three-dimensional system according to the structure of limit sets" Diff. equations, T.VII. 1971. No. 3. pp. 461 – 470.
2. Khaimov N.B. Some theorems on singular points of the first group. Uch. Zap. Stalinabdsky pedins-that vol. I, 1952 pp. 45 - 113
3. Khusanov B., Sharipov Sh.R. Mamashev S.Kh. "On some questions of the trajectory of an n-dimensional generalized homogeneous differential system". Ed. AZh KazSSR. Series of physical and mathematical sciences, 1981, No. 3. pp 33 – 43.
4. Sharipov Sh.R. "On the behavior of integral curves of a system of n - equations at infinity" Scientific works. Departments of higher mathematics. MKI. 1968, pp. 103-127.
5. Poincare A. "On curves defined by differential equations". M - L. GTTI. 1947 – 390 pages